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Coefficient of restitution and linear dashpot model revisited

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Abstract With the assumption of a linear-dashpot interaction force, we compute the coefficient of restitution as a function of the elastic and dissipative material constants, k and γ by integrating Newton's equation of motion for an isolated pair of colliding particles. It turns out that an expression $\varepsilon(k, \gamma)$ widely used in the literature is even qualitatively incorrect due to an incorrect definition of the duration of the collision.

Keywords particle collisions · coefficient of restitution

1 Introduction

The (normal) coefficient of restitution of colliding spherical particles relates the post-collisional velocities \mathbf{v}'_i and \mathbf{v}'_j and the corresponding pre-collisional velocities \mathbf{v}_i and \mathbf{v}_j ,

$$\begin{aligned}\mathbf{v}'_i &= \mathbf{v}_i - \frac{1 + \varepsilon}{2} [(\mathbf{v}_i - \mathbf{v}_j) \cdot \mathbf{e}_{ij}] \mathbf{e}_{ij} \\ \mathbf{v}'_j &= \mathbf{v}_j + \frac{1 + \varepsilon}{2} [(\mathbf{v}_i - \mathbf{v}_j) \cdot \mathbf{e}_{ij}] \mathbf{e}_{ij}\end{aligned}\quad (1)$$

where

$$\mathbf{e}_{ij} \equiv \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|} \quad (2)$$

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is the unit vector at the instance of the collision. For simplicity, we consider particles of identical mass - the generalization to unequal masses is straightforward.

The coefficient of restitution is the most fundamental quantity in the theory of dilute granular systems. This coefficient together with the assumption of *molecular chaos* is the foundation of the Kinetic Theory of granular gases, based on the Boltzmann or Enskog kinetic equation, see, e.g., [1; 7; 8] and many references therein. On the other hand, the coefficient of restitution is also essential for event-driven Molecular Dynamics simulations which allow for a significant speed-up as compared to traditional force-based simulations, see, e.g., [9] for a detailed discussion. The most important precondition of Eq. (1) is the assumption of exclusively binary collision, that is, the system is assumed to be dilute enough such that multiple-particle contacts can be neglected. The latter condition seems to be rather restrictive, however, it was shown in many simulations that event-driven simulations are applicable up to rather high density [6].

The coefficient of restitution is, however, not a fundamental material or particle property. Instead, the particle interaction forces and Newton's equation of motion govern the dynamics of a mechanical many-particle system. Therefore, if we wish to use the concept of the coefficient of restitution and the simple equation (1) to compute the dynamics of a granular system, we have to assure that Eq. (1) yields the same post-collisional velocities as Newton's equation of motion would do.

2 Coefficient of restitution and Newton's equation of motion for frictionless spherical particles

Let us consider the relation between the coefficient of restitution and the solution of Newton's equation of motion. Assume two approaching particles at velocities \mathbf{v}_1 and \mathbf{v}_2 come into contact at time $t = 0$. From this instant on they deform one another until the contact is lost at time $t = t_c$. We shall consider the end of the collision at time t_c separately below. For sufficiently short collisions we can treat the unity vector \mathbf{e} as a constant. Then we can describe the collision by the

mutual deformation

$$\xi(t) \equiv \max(0, 2R - |\mathbf{r}_1 - \mathbf{r}_2|) \quad (3)$$

where $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ are the time-dependent positions of the particles and R is their radius. The interaction force between the particles is model specific, e.g. [9; 11], and, in general, a function of the material parameters, the particle masses and radii, the deformation ξ and the deformation rate $\dot{\xi}$. Its time dependence is expressed via $F = F(\xi(t), \dot{\xi}(t))$. Newton's equation of motion for the colliding particles reads

$$\frac{m}{2}\ddot{\xi} + F(\dot{\xi}, \xi) = 0; \quad \xi(0) = 0; \quad \dot{\xi} = v \quad (4)$$

where v is the normal component of the initial relative velocity,

$$v \equiv \frac{[\mathbf{v}_1(0) - \mathbf{v}_2(0)] \cdot [\mathbf{r}_1(0) - \mathbf{r}_2(0)]}{2R}. \quad (5)$$

The coefficient of restitution follows from the solution of Eq. (4),

$$\varepsilon = -\dot{\xi}(t_c)/v, \quad (6)$$

where t_c is the duration of the collision.

Thus, the coefficient of restitution due to a specified interaction force law can be determined by solving the equation of motion [10; 11; 13]. In general, ε is a function of the material parameters, the particle masses and radii and the impact velocity v .

3 Linear dashpot model

Consider the *linear dashpot* force,

$$F(\xi, \dot{\xi}) = -k\xi - \gamma\dot{\xi}. \quad (7)$$

This force is problematic as a particle interaction model since particles made of a linear-elastic material do not reveal a linear repulsive force, neither in three dimensions [4] nor in 2D [3]. The subsequent analysis can be also performed for more realistic force models, such as viscoelastic forces in 3d [2] and 2d [12] which will be published elsewhere.

The linear dashpot model is of interest here because this force leads to a *constant* coefficient of restitution, e.g. [11], that is, ε does not depend on the impact velocity but only on the material parameters k and γ . The constant coefficient of restitution in turn is the preferred model in both the Kinetic Theory of granular gases and also in event-driven simulations of granular matter. In contrast, more realistic force models lead to an impact-velocity dependent coefficient of restitution [5; 10; 13; 14; 15].

The equation of motion corresponding to Eq. (7)

$$m\ddot{\xi} + 2\gamma\dot{\xi} + 2k\xi = 0 \quad (8)$$

with initial conditions

$$\xi(0) = 0; \quad \dot{\xi}(0) = v \quad (9)$$

has two solutions. With the abbreviations

$$\omega_0^2 \equiv \frac{2k}{m}; \quad \beta \equiv \frac{\gamma}{m}; \quad \omega \equiv \sqrt{\omega_0^2 - \beta^2}. \quad (10)$$

we can write the solution for the case of low damping ($\beta < \omega_0$) as

$$\xi(t) = \frac{v}{\omega} e^{-\beta t} \sin \omega t, \quad (11)$$

while for the case of high damping ($\beta > \omega_0$) we have

$$\xi(t) = \frac{v}{\Omega} e^{-\beta t} \sin \Omega t, \quad (12)$$

with

$$\Omega = \sqrt{\beta^2 - \omega_0^2}. \quad (13)$$

To determine the coefficient of restitution according to Eq. (6) we need the duration of the collision t_c . A natural choice seems to be

$$\xi(t_c^0) = 0; \quad t_c^0 > 0, \quad (14)$$

that is, the collision is finished when $|\mathbf{r}_i - \mathbf{r}_j| = 2R$. With this condition we obtain $t_c^0 = \pi/\omega$ and the coefficient of restitution

$$\varepsilon_d^0 \equiv \frac{\dot{\xi}(t_c^0)}{\dot{\xi}(0)} = e^{-\beta\pi/\omega}, \quad (15)$$

for the case of low damping which is widely used, e.g. in [11] and many others.

Unfortunately, condition (14) cannot be correct since at time $t_c^0 = \pi/\omega$ the interaction force is negative,

$$m\ddot{\xi}\left(\frac{\pi}{\omega}\right) = -\frac{2m\gamma\beta}{\omega} e^{-\beta\pi/\omega} < 0, \quad (16)$$

see Fig. 1, which contradicts the assumption of exclusively repulsive interaction between granular particles. Moreover, for the case of high damping Eq. (14) has no real solution, that is, the particles stick together after a head-on collision (see Fig. 2). One can call this situation *dissipative capture*.

To avoid these artifacts, instead of Eq. (14), we now explicitly take into account the condition of no attraction. That means, instead of Eq. (7) we have to use the force $F^* = \max(0, F)$, with F given by Eq. (7). This expression as is in common use in (force-based) Molecular Dynamics simulations, assures that only repulsive forces act. It is clear that once the force becomes zero, the collision is finished,

$$\dot{\xi}(t_c) = 0; \quad t_c > 0; \quad \dot{\xi}(t_c) < 0, \quad (17)$$

that is, the collision is complete when the repulsive force between the particles vanishes and the surfaces of the particles

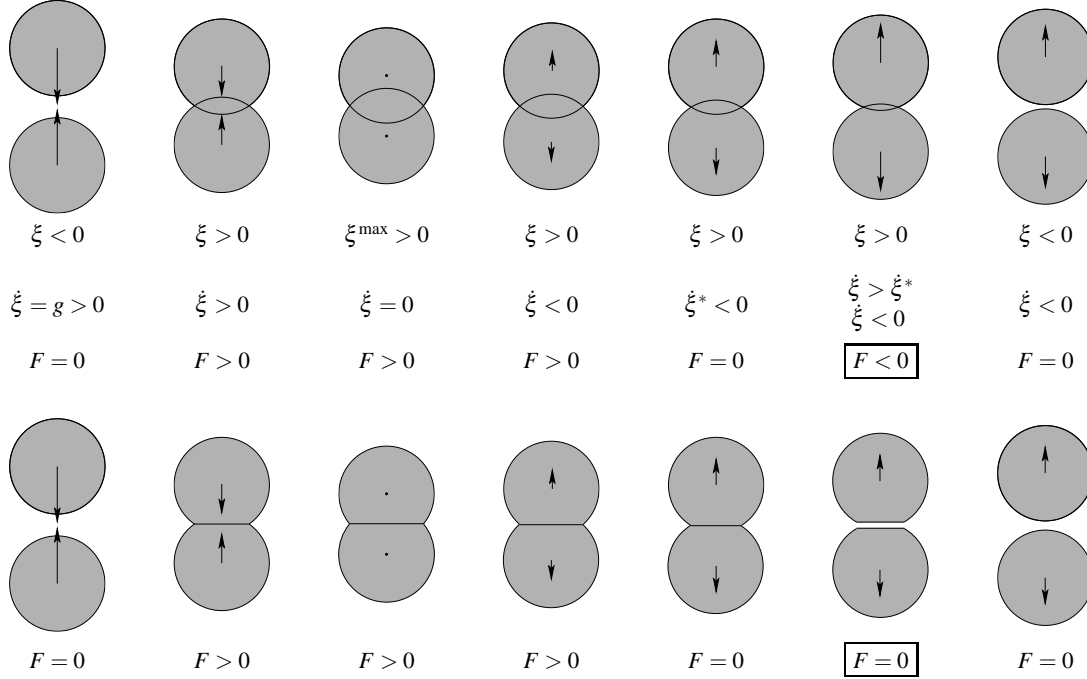


Fig. 1 Sketch of a particle collision. The upper part of the figure visualizes the condition (14) for the end of the collision. As an artifact of the model, there appear attracting forces towards the end of the collision (boxed notation). Lower part: The condition (17) for the end of the collision avoids this artifact.

separate from one another. This condition takes, thus, into account that Eq. (7) applies to *particles in contact*.

For the case of low damping Eqs. (11) and (17) yield

$$\tan \omega t_c = -\frac{2\beta\omega}{\omega^2 - \beta^2}. \quad (18)$$

One has to take care to select the correct branch of the arc tangent for $\beta < \omega_0/\sqrt{2}$ (or $\beta < \omega$) and for $\beta > \omega_0/\sqrt{2}$. The solution reads

$$t_c = \begin{cases} \frac{1}{\omega} \left(\pi - \arctan \frac{2\beta\omega}{\omega^2 - \beta^2} \right) & \text{for } \beta < \frac{\omega_0}{\sqrt{2}} \\ \frac{1}{\omega} \arctan \frac{2\beta\omega}{\omega^2 - \beta^2} & \text{for } \beta > \frac{\omega_0}{\sqrt{2}} \end{cases} \quad (19)$$

For the coefficient of restitution we find

$$\varepsilon_d = e^{-\beta t_c} \quad (20)$$

regardless of the branch of the solution of Eq. (18). Together with the solution for the case of high damping which can be obtained by straight-forward computations we obtain the coefficient of restitution:

$$\varepsilon_d = \begin{cases} \exp \left[-\frac{\beta}{\omega} \left(\pi - \arctan \frac{2\beta\omega}{\omega^2 - \beta^2} \right) \right] & \text{for } \beta < \frac{\omega_0}{\sqrt{2}} \\ \exp \left[-\frac{\beta}{\omega} \arctan \frac{2\beta\omega}{\omega^2 - \beta^2} \right] & \text{for } \beta \in \left[\frac{\omega_0}{\sqrt{2}}, \omega_0 \right] \\ \exp \left[-\frac{\beta}{\Omega} \ln \frac{\beta + \Omega}{\beta - \Omega} \right] & \text{for } \beta > \omega_0 \end{cases}$$

(21)

In accordance with physical intuition we have $\varepsilon_d > \varepsilon_d^0$ since the result given in Eq. (21) does not suffer from the artifact of unphysical dissipative capture does not occur – even for large damping constant the particles separate eventually. For very high damping the coefficient of restitution can be approximated as

$$\varepsilon_d \approx \frac{\omega_0^2}{4\beta^2} \quad \text{for } \beta \gg \omega_0 \quad (22)$$

Fig. 2 shows both the (wrong) coefficient of restitution as derived from criterion (14) (dashed line) and the coefficient of restitution derived from the correct criterion (17) (full line). As the solution Eq. (21) only depends on the ratio $\alpha = \beta/\omega_0$ we draw the coefficient of restitution as a function of this parameter. Figure 2 shows that the unphysical attraction indeed yields a too low coefficient of restitution. The inset shows a logarithmic plot of the same curve showing that there is no capture if the correct criterion is applied.

4 Conclusion

Assuming a linear dashpot interaction force we computed ε as a function of the elastic and dissipative material properties by integrating Newton's equation of motion for a pair of colliding particles. The condition for the duration t_c of the collision $\xi(t_c) = 0$, that is, the interaction stops when the

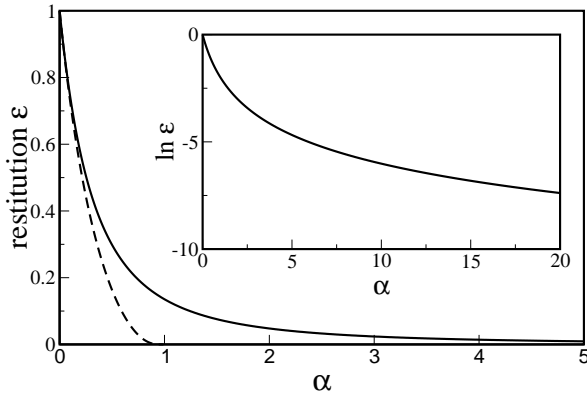


Fig. 2 Coefficient of restitution as function of $\alpha = \beta/\omega_0$. The dashed curve is the result for the wrong termination condition $\xi(t_c^0) = 0$. Obviously, for high damping $\alpha > 1$ we have dissipative capture. The inlay shows $\ln \epsilon$ as a function of α for the correct termination condition. There is no dissipative capture even for very high damping.

particles have the distance $|\mathbf{r}_i - \mathbf{r}_j| = 2R$, is widely used in the literature, however, it leads to erroneous attracting forces between the particles and, thus, to wrong values for the coefficient of restitution. In extreme cases, the result may become even qualitatively incorrect since despite the interaction force being purely repulsive, the particles may agglomerate. In this state one would need to supply energy to separate the particles from one another.

To account for the fact that the interaction force can never become attractive, we use the condition $\xi(t_c) = 0$ which avoids the mentioned artifacts. This condition takes into account that the collision may be completed even before $\xi = 0$, that is, the surfaces of the particles lose contact slightly before the distance of their centers exceeds the sum of their radii. Thus, the deformation of the particles may last longer than the time of contact and the particles gradually recover their spherical shape only *after* they lost contact.

The resulting coefficient of restitution is always larger than the value based on $\xi(t_c^0) = 0$. In agreement with the repulsive character of the interaction force, the coefficient of restitution is always well defined and non-zero.

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